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A scalar form of the complementary mild-slope equation

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Mild-slope (MS) type equations are depth-integrated models, which predict under appropriate conditions refraction and diffraction of linear time-harmonic water waves. Among these equations, the complementary mild-slope equation (CMSE) was shown to give better agreement with exact two-dimensional linear theory compared to other MS-type equations. Nevertheless, it has a disadvantage of being a vector equation, i.e. it requires solving a system of two coupled partial differential equations. In addition, for three-dimensional problems, there is a difficulty in constructing the additional boundary condition needed for the solution. In the present work, it is shown how the vector CMSE can be transformed into an equivalent scalar equation using a pseudo-potential formulation. The pseudo-potential mild-slope equation (PMSE) preserves the accuracy of the CMSE while avoiding the need of an additional boundary condition. Furthermore, the PMSE significantly reduces the computational effort relative to the CMSE, since it is a scalar equation. The accuracy of the new model was tested numerically by comparing it to laboratory data and analytical solutions.

Key words: surface gravity waves

1. Introduction

Surface water wave problems are governed, under plausible assumptions, by the Laplace equation together with free-surface, bottom and lateral boundary conditions. The three-dimensional problem is complicated to model, and its solution requires significant computational effort. Hence, a reduction of one dimension is done in many applications in order to reduce the numerical complications and costs.

A commonly employed approach for this reduction yields mild-slope (MS) type equations, which apply to linear time-harmonic problems. The origin of this discipline is the mild-slope equation (MSE) by Berkhoff (1972). The MSE and other MS-type models achieve this reduction by assuming a vertical profile and then averaging the governing equation over the depth, which results in the elimination of the vertical coordinate.

Many works continued Berkhoff's pioneering derivation. Specifically, the extension of this model to hold for steep slopes and curvatures was addressed. Rapid undulations were considered (see Kirby 1986), and the MS assumptions were relaxed (see Chamberlain & Porter 1995) to hold the higher-order bottom components $\nabla^2 h$ and $(\nabla h)^2$. Furthermore, in order to increase the ability to deal with steep slopes, there

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were as well improvements of the fulfilment of the bottom boundary condition (e.g. Hsu *et al.* 2006; Chandrasekera & Cheung 2001 and Porter & Porter 2006). These improved MS-type models were commonly applied; simplified for use (e.g. Lee *et al.* 1998); investigated analytically (see Agnon 1999) and even extended to more complex physical problems (e.g. Silva, Salles & Palacio 2002).

Nevertheless, in all of the above mentioned models the only mode considered is the free wave mode. In order to further improve the results, a set of evanescent modes were added to the MS-type model (see Massel 1993 and Porter & Staziker 1995). This complicates the model, as it requires solving a coupled set of equations for the various modes. But, it yields a more complete representation of the flow field, which results in a superior accuracy. This approach was also developed and extended to yield even higher accuracy (see Athanassoulis & Belibassakis 1999; Belibassakis, Athanassoulis & Gerostathis 2001; Chamberlain & Porter 2006).

In contrast to the above formulations, Kim & Bai's (2004) complementary mild-slope equation (CMSE) is derived in terms of a stream function vector rather than a velocity potential, and consists of one vertical profile of the free wave mode. Unlike the vertical profile in the velocity potential formulations, the vertical profile in the stream function formulation satisfies exactly the kinematic boundary condition on uneven bottoms. This leads to better agreements with the exact linear theory compared to other MS-type equations for two-dimensional problems (see Kim & Bai 2004).

Unfortunately, in the three-dimensional case, the stream function vector formulation complicates the governing equation by making it a two-dimensional vector equation rather than a scalar one. This increases the computational effort and complexity, but this is not the only difficulty in using the CMSE for solving three-dimensional problems. A set of two coupled differential equations requires two lateral boundary conditions. Still, by using the impermeable boundary condition only one relation can be constructed. This leads to a deficiency of one lateral boundary condition, which Kim & Bai did not refer to.

These difficulties led us to construct a scalar model that still maintains the advantages of the stream function formulation. Deriving this scalar model is the main goal of the present work. The plan is to present the CMSE in § 2, and to derive its alternative vector formulation in § 3 as the first step toward a scalar MS-type model. A pseudo-potential definition and the pseudo-potential mild slope equation (PMSE) scalar model, which is the main result of this work, are presented in § 4. The lateral boundary condition for the PMSE in terms of the pseudo-potential is discussed in § 6, and numerical simulations are presented in § 7.

2. The complementary mild-slope equation

Let us consider a time-harmonic linear wave motion over a variable bottom topography, $z=-h(x, y)$, and define $\boldsymbol{\Psi}$ as a stream function vector

$$\boldsymbol{\Psi}(\boldsymbol{x}, z) \equiv \begin{pmatrix} \Psi^{I} \\ \Psi^{II} \end{pmatrix} \equiv \int_{-h}^{z} \boldsymbol{u}(\boldsymbol{x}, \zeta)\,\mathrm{d}\zeta, \qquad \boldsymbol{u} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}. \tag{2.1}$$

Here, $\boldsymbol{x}$ is the horizontal position vector, $Re[\boldsymbol{u}\mathrm{e}^{-\mathrm{i}\omega t}]$ is the horizontal velocity vector and ω is the angular frequency of the incoming wave. Using (2.1) the velocity field and the linear free-surface elevation are found from $\boldsymbol{\Psi}$ by:

$$\boldsymbol{u} = \frac{\partial \boldsymbol{\Psi}}{\partial z}, \quad w = -\nabla \cdot \boldsymbol{\Psi}, \tag{2.2}$$

$$\eta = \frac{1}{\mathrm{i}\omega} \nabla \cdot \boldsymbol{\Psi}_0, \tag{2.3}$$

where $Re[w\,\mathrm{e}^{-\mathrm{i}\omega t}]$ is the vertical velocity, $Re[\eta\,\mathrm{e}^{-\mathrm{i}\omega t}]$ is the surface elevation, $\boldsymbol{\Psi}_0$ is $\boldsymbol{\Psi}$ on the still water level ($z=0$) and ∇ is the horizontal vector differential operator. Kim & Bai (2004) formulated, as well, the linear time-averaged Lagrangian density for this problem in terms of $\boldsymbol{\Psi}$,

$$L = \frac{1}{2}\rho \int\int\int_{-h(x,y)}^{0} \left\{ |\nabla \cdot \boldsymbol{\Psi}|^2 + \left|\frac{\partial \boldsymbol{\Psi}}{\partial z}\right|^2 \right\} \mathrm{d}z - \frac{\rho g}{2\omega^2} |\nabla \cdot \boldsymbol{\Psi}_0|^2 . \tag{2.4}$$

In order to eliminate the z-coordinate and construct a MS-type equation they assumed a vertical profile

$$\boldsymbol{\Psi}(\boldsymbol{x}, z, t) = f(k, h, z)\boldsymbol{\Psi}_0(\boldsymbol{x}, t), \quad f(k, h, z) = \frac{\sinh(k(h)(z+h))}{\sinh(k(h)h)}. \tag{2.5}$$

The vertical profile assumption (2.5) was applied to the Lagrangian (2.4), and a first variation with respect to $\boldsymbol{\Psi}_0$ was taken to yield the CMSE:

$$-\nabla\left(a\left(\nabla \cdot \boldsymbol{\Psi}_0\right) + b\left(\nabla h \cdot \boldsymbol{\Psi}_0\right)\right) + \left(b\nabla \cdot \boldsymbol{\Psi}_0 + c\nabla h \cdot \boldsymbol{\Psi}_0\right)\nabla h - k(h)^2 a \boldsymbol{\Psi}_0 = 0. \tag{2.6}$$

The coefficients a, b and c are defined by

$$a(h) = \int_{-h}^{0} f^2 \mathrm{d}z - \frac{g}{\omega^2} = -\frac{\coth(kh)}{2k}\left(1 + \frac{2kh}{\sinh(2kh)}\right) = -\frac{gk^2}{\omega^4} CC_g, \tag{2.7}$$

$$b(h) = \int_{-h}^{0} f \frac{\partial f}{\partial h} \mathrm{d}z = \frac{1}{4\sinh^2(kh)} \frac{2kh\cosh(2kh) - \sinh(2kh)}{2kh + \sinh(2kh)}, \tag{2.8}$$

$$\begin{aligned} c(h) &= \int_{-h}^{0} \left(\frac{\partial f}{\partial h}\right)^2 \mathrm{d}z \\ &= \frac{k}{12\sinh^2(kh)} \frac{-12kh + 8(kh)^3 + 3\sinh(4kh) + 12(kh)^2\sinh(2kh)}{(2kh + \sinh(2kh))^2}. \end{aligned} \tag{2.9}$$

Note that the CMSE is presented here with a minor correction. The c-term in equation (2.6) is the coefficient of $(\nabla h \cdot \boldsymbol{\Psi}_0)\nabla h$ and not $(\nabla h \cdot \nabla h)\boldsymbol{\Psi}_0$ as was given by Kim & Bai (2004). This correction makes a difference only for three-dimensional problems and does not change the two-dimensional equation. Kim & Bai's (2004) numerical simulations have been conducted only for two-dimensional problems, and therefore, their results are valid.

3. An alternative vector formulation of the CMSE

The case of a discontinuous bottom slope requires special consideration as it consists of infinite curvature spikes. Kim & Bai (2004) discussed the jump condition required for the CMSE in this case. In order to enable for solving without applying jump conditions, Porter (2003) presented a method that transforms the modified mild-slope equation (MMSE) to an alternative form containing only first-order bottom derivatives. The basic concept of Porter's method is to write the potential as a product of two functions: a scaling function and a scaled potential. The scaling function is a function that depends only on the bottom profile and is defined specifically to cancel out the term containing the bottom curvature. In this section, the same technique is applied to the vector CMSE.

Let us define an auxiliary vector $\boldsymbol{\varphi}$ such that

$$\boldsymbol{\Psi}_0(\boldsymbol{x}) = s(h)\boldsymbol{\varphi}(\boldsymbol{x}), \quad \boldsymbol{\varphi} = \begin{pmatrix} \varphi^I \\ \varphi^{II} \end{pmatrix}, \tag{3.1}$$

and apply this definition to equation (2.6) to yield

$$s(h)\nabla(a(\nabla\cdot\boldsymbol{\varphi})) + \nabla((as'(h) + bs(h))(\nabla h\cdot\boldsymbol{\varphi})) + k(h)^2 as(h)\boldsymbol{\varphi} \\ + (s'(h)a\nabla\cdot\boldsymbol{\varphi} - s(h)b\nabla\cdot\boldsymbol{\varphi} - (s'(h)b + s(h)c)(\nabla h\cdot\boldsymbol{\varphi}))\nabla h = 0. \tag{3.2}$$

The additional function allows to arbitrarily impose a desirable condition. This condition is selected in the same manner of Porter (2003) as follows:

$$as'(h) + bs(h) = 0. \tag{3.3}$$

The solution for $s'(h)$ from equation (3.3) can be used to eliminate $s'(h)$ from equation (3.2). By using the relation $\nabla a = 2b\nabla h$ and further dividing by $s(h)$, equation (3.2) yields an alternative form of the CMSE

$$\nabla\,(\nabla\cdot\boldsymbol{\varphi}) + k^2\boldsymbol{\varphi} + \alpha\,(\nabla h\cdot\boldsymbol{\varphi})\,\nabla h = 0, \tag{3.4}$$

where

$$\alpha(h) = \frac{b(h)^2 - a(h)c(h)}{a(h)^2}. \tag{3.5}$$

Equation (3.4) contains only first derivatives of h, and can be used to solve $\boldsymbol{\varphi}$ for non-smooth bathymetries. By further investigation of equation (3.4), we can see that it contains no derivatives of $\boldsymbol{\varphi}$ except for the $\nabla\nabla\cdot$ operator. For our purpose, this is the main contribution of definition (3.1), as it will enable formulating the CMSE as a scalar equation.

The scaling function $s(h)$ can be found analytically. By applying the integrating factor technique to equation (3.3) the solution takes the form

$$s(h) = \mathrm{e}^{-\int (a/b)\mathrm{d}h}. \tag{3.6}$$

The integral can be solved by changing the integration argument from h to kh using the relation

$$\mathrm{d}h = \frac{\mathrm{d}(kh)}{k'(h)h + k(h)}, \tag{3.7}$$

where $k'(h)$ is expressed by taking the derivative of the linear dispersion relation to yield

$$s(h) = \sqrt{\frac{\sinh(2k(h)h)}{2k(h)h + \sinh(2k(h)h)}}. \tag{3.8}$$

4. A scalar formulation for the CMSE

For three-dimensional problems the CMSE (and also its alternative form given in §3) is essentially a coupled set of two differential equations, which is written in a vector stream function formulation. Let us reduce the CMSE to a scalar equation in term of a new 'pseudo-potential':

$$\Omega \equiv \nabla \cdot \boldsymbol{\varphi}. \tag{4.1}$$

We start by rewriting the alternative form of the CMSE (3.4) using matrix notation

$$\boldsymbol{B}\boldsymbol{\varphi} = -\nabla(\nabla \cdot \boldsymbol{\varphi}) \tag{4.2}$$

where,

$$\boldsymbol{B} = \begin{pmatrix} k^2 + \alpha h_x^2 & \alpha h_x h_y \\ \alpha h_x h_y & k^2 + \alpha h_y^2 \end{pmatrix}. \tag{4.3}$$

Applying the pseudo-potential definition (4.1) to equation (4.2) yields

$$\boldsymbol{B}\boldsymbol{\varphi} = -\nabla \Omega. \tag{4.4}$$

In order to write equation (4.4) solely in terms of Ω, it will be multiplied by $\boldsymbol{B}^{-1}$ and afterward by the operator $\nabla \cdot$. This leads to the PMSE:

$$\nabla \cdot \left(\boldsymbol{B}^{-1} \nabla \Omega\right) + \Omega = 0, \tag{4.5}$$

$$\boldsymbol{B}^{-1} = \begin{pmatrix} \dfrac{k^2 + \alpha h_y^2}{k^2\left(k^2 + \alpha h_x^2 + \alpha h_y^2\right)} & -\dfrac{\alpha h_x h_y}{k^2\left(k^2 + \alpha h_x^2 + \alpha h_y^2\right)} \\ -\dfrac{\alpha h_x h_y}{k^2\left(k^2 + \alpha h_x^2 + \alpha h_y^2\right)} & \dfrac{k^2 + \alpha h_x^2}{k^2\left(k^2 + \alpha h_x^2 + \alpha h_y^2\right)} \end{pmatrix}.$$

Equation (4.5) is a scalar equation rather than a vector equation as the CMSE. By applying the operator $s\boldsymbol{B}^{-1}$ to equation (4.4) together with (3.1), $\boldsymbol{\Psi}_0$ can be expressed as

$$\boldsymbol{\Psi}_0 = -s\boldsymbol{B}^{-1} \nabla \Omega, \tag{4.6}$$

which enables its reconstruction after solving for Ω.

5. Parabolic approximation

For various problems, in which the reflected waves can be neglected, it is plausible to assume a progressive wave field to the first order. The wave amplitude can contain the smaller deviations from the first order wave field. The resulting equation becomes parabolic instead of elliptic. This enables to extensively reduce the computer storage and CPU time needed for the numerical solution as the wave flow problem can be solved as a moving front. There are various ways of applying parabolic approximations. In this section the parabolic approximation will be formulated for the PMSE following a method used by Kaihatu & Kirby (1995).

Let us assume the problem to have the form of a progressive wave field, so the pseudo-potential has the following behavior

$$\Omega = A(x, y) \mathrm{e}^{\mathrm{i} \int k(x,y)\, \mathrm{d}x}. \tag{5.1}$$

Here, A is a complex function slowly varying in x and y that represents the stream function complex amplitude. Applying equation (5.1) to the building blocks of equation (4.5) yields the relations:

$$\left.\begin{aligned}\Omega_x &= (A_x + \mathrm{i}kA)\mathrm{e}^{\mathrm{i}\int k\mathrm{d}x},\\ \Omega_y &= [A\mathrm{e}^{\mathrm{i}\int k\,\mathrm{d}x}]_y,\\ \Omega_{xx} &= (A_{xx} + 2\mathrm{i}kA_x + \mathrm{i}k_xA - k^2A)\mathrm{e}^{\mathrm{i}\int k\mathrm{d}x},\\ \Omega_{xy} &= [(A_x + \mathrm{i}kA)\mathrm{e}^{\mathrm{i}\int k\,\mathrm{d}x}]_y,\\ \Omega_{yy} &= [A\mathrm{e}^{\mathrm{i}\int k\,\mathrm{d}x}]_{yy}.\end{aligned}\right\} \tag{5.2}$$

The wave is assumed to be propagating mostly along the positive x-direction. It consists of rapid variation accounted for by the complex exponential given in equation (5.1). Following Yue & Mei (1980), we scale the derivatives of A as follows

$$\frac{\partial A}{\partial x} = O(\epsilon^2), \qquad \frac{\partial A}{\partial y} = O(\epsilon). \tag{5.3}$$

By assuming that the behavior in the x-direction is mostly accounted for by the complex exponential, and by applying the ordering stated in (5.3), we can neglect the higher-order term A_{xx}. This changes the equation's nature from elliptic to parabolic.

In order to factor out the dependence of k in the y-direction, a y-averaged wave number function $\bar{k}(x)$ can be defined as a reference (see Lozano & Liu 1980),

$$\Omega = a(x, y)\mathrm{e}^{\mathrm{i}\int \bar{k}(x)\,\mathrm{d}x}. \tag{5.4}$$

Using equations (5.1) and (5.4), a relation between $A(x, y)$ and $a(x, y)$ can be constructed as

$$A(x, y) = a(x, y)\mathrm{e}^{\mathrm{i}\int \bar{k}(x)\,\mathrm{d}x - \mathrm{i}\int k(x,y)\,\mathrm{d}x}. \tag{5.5}$$

Substituting equation (5.5) to equation set (5.2) yields

$$\left.\begin{aligned}\Omega_x &= (a_x + \mathrm{i}\bar{k}a)\mathrm{e}^{\mathrm{i}\int \bar{k}\,\mathrm{d}x},\\ \Omega_y &= a_y e^{\mathrm{i}\int \bar{k}\,\mathrm{d}x},\\ \Omega_{xx} &= (2\mathrm{i}ka_x + \mathrm{i}k_xa - 2\bar{k}ka + k^2a)\mathrm{e}^{\mathrm{i}\int \bar{k}\mathrm{d}x},\\ \Omega_{xy} &= (a_{xy} + \mathrm{i}\bar{k}a_y)\mathrm{e}^{\mathrm{i}\int \bar{k}\,\mathrm{d}x},\\ \Omega_{yy} &= a_{yy}\mathrm{e}^{\mathrm{i}\int \bar{k}\,\mathrm{d}x}.\end{aligned}\right\} \tag{5.6}$$

Equation (5.4) and equation set (5.6) together with equation (4.5) form a parabolic equation in terms of the slowly varying wave amplitude $a(x, y)$.

6. Lateral boundary conditions

In order to formulate a complete mathematical representation of a wave flow problem, lateral boundary conditions should be supplied in terms of the pseudo-potential together with the PMSE (4.5). For the stream function formulation the impermeable boundary condition is a Dirichlet boundary condition that defines the boundary as a stream line. This concept can be written in the following way,

$$\boldsymbol{\Psi}_0 \cdot \hat{\boldsymbol{n}} = 0, \tag{6.1}$$

where $\hat{\boldsymbol{n}}$ denotes the outward unit normal vector at the boundary. By using equation (4.6), it is easy to express the impermeable boundary condition in terms of the pseudo-potential Ω as

$$(\boldsymbol{B}^{-1}\nabla\Omega) \cdot \hat{\boldsymbol{n}} = 0. \tag{6.2}$$

From equation (6.2) it can be seen that the pseudo-potential formulation of the impermeable lateral boundary condition resembles more to the Neumann boundary condition of the velocity potential formulation.

7. Numerical results

In two-dimensional problems, the CMSE and the PMSE are analytically similar. Therefore, the superior numerical results presented in Kim & Bai (2004) for the CMSE hold as well for the PMSE. Some additional numerical calculations are presented in this section to reassure the use of the PMSE in three-dimensional problems as well. For the numerical integration the NDSolve function of the MATHEMATICA 6 software was used.

The first numerical simulation is of obliquely incident waves propagating on a sloped beach. This is a quasi-three-dimensional problem. If we set the coordinate system so that the bottom changes only in the x-direction, the wave component does not change its wave number in the y-direction and the formulating functions can be written as

$$\begin{pmatrix} \varOmega(x, y) \\ \boldsymbol{\varPsi}_0(x, y) \\ \varPhi(x, y) \end{pmatrix} = \begin{pmatrix} \bar{\varpi}(x) \\ \psi(x) \\ \phi(x) \end{pmatrix} \mathrm{e}^{\mathrm{i}k_y y}. \tag{7.1}$$

The case of 45° incidence angle waves on a 45° plane beach was chosen for this simulation. The numerical simulations of the PMSE, the CMSE and the MMSE (Chamberlain & Porter 1995) together with an exact analytical solution of Ehrenmark (1998) are given in figure 1. It can be seen that all of the models have good agreements to the analytical solution with an advantage to the ones of the PMSE and the CMSE.

The second numerical simulation is of monochromatic waves propagating on a flat bottom with a circular shoal area. The chopped sphere underwater sea mount acts as a lens that focuses the waves and creates a cusped caustic. The simulations will be compared to a wave tank experiment conducted by Ito & Tanimoto (1972). The bathymetry of the experiment and the monitored sections are shown in figure 2. The constant bottom depth surrounding the shoal is $h_0 = 0.15$ m. The centre of the circular shoal area is located at $(x_c, y_c) = (1.2\text{ m}, 1.2\text{ m})$ giving the shallowest water depth as 0.05 m. The water depth at the shoal area ($r \leqslant 0.8$ m) is defined as

$$h(x, y) = 0.05\text{ m} + 0.15625\text{ m}^{-1}((x - x_c)^2 + (y - y_c)^2),$$

and the wave height and period are H = 1.04 cm and T = 0.511 sec.

A parabolic approximation was applied to the models in order to allow for a less elaborate simulation. For the MMSE the parabolic approximation was applied in the same manner as it was done by Kaihatu & Kirby (1995) for the MSE, and for the PMSE it was applied as was presented in § 5. The numerical simulations of the models together with the wave gauge measurements of Ito & Tanimoto (1972) for this experiment are given in figure 3. Note that the results of Ito & Tanimoto (1972) fail to be symmetric as expected from the symmetry of the experimental set-up. Therefore, the wave gauge measurements in cross-sections 2 and 3 are duplicated in a mirror image, since asymmetry is due to experimental errors. The original data is marked in solid circles, and the mirror image is marked in solid squares. It can be seen that both the PMSE and the MMSE models have good agreements with the measured data.

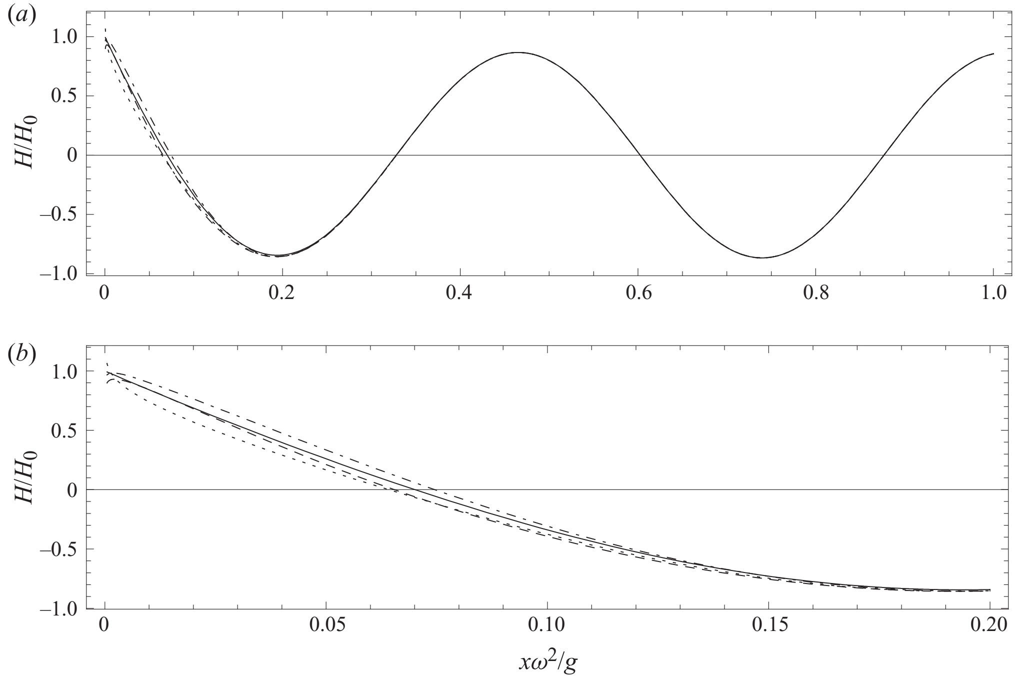

FIGURE 1. (*a*) The normalized wave height of the PMSE (dashed), CMSE (dot-dashed) and the MMSE (dotted) with respect to the exact analytical solution of Ehrenmark (1998) (solid) for a 45° incidence angle waves on a 45° plane beach. (*b*) A magnification of the shallow part in (*a*) starting from $kh \approx 0.46$.

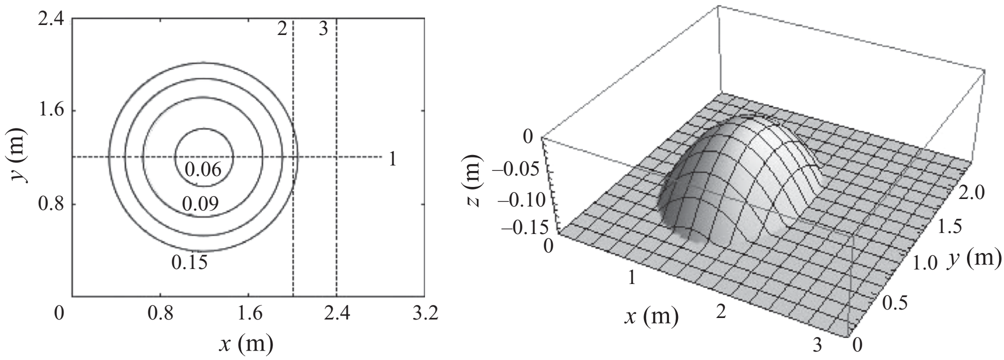

FIGURE 2. The bathymetry in the experiment of Ito & Tanimoto (1972). The wave maker is positioned at $x = 0$. Dashed lines indicate the cross-sections monitored by wave gauges. All units are stated in meters.

8. Summary and conclusions

A new scalar MS-type equation was derived from the vector CMSE. This equation is named the pseudo-potential mild-slope equation. It allows solving for three-dimensional problems and produces accurate results with much less computational effort than the CMSE. Numerical simulations of the model were compared with an

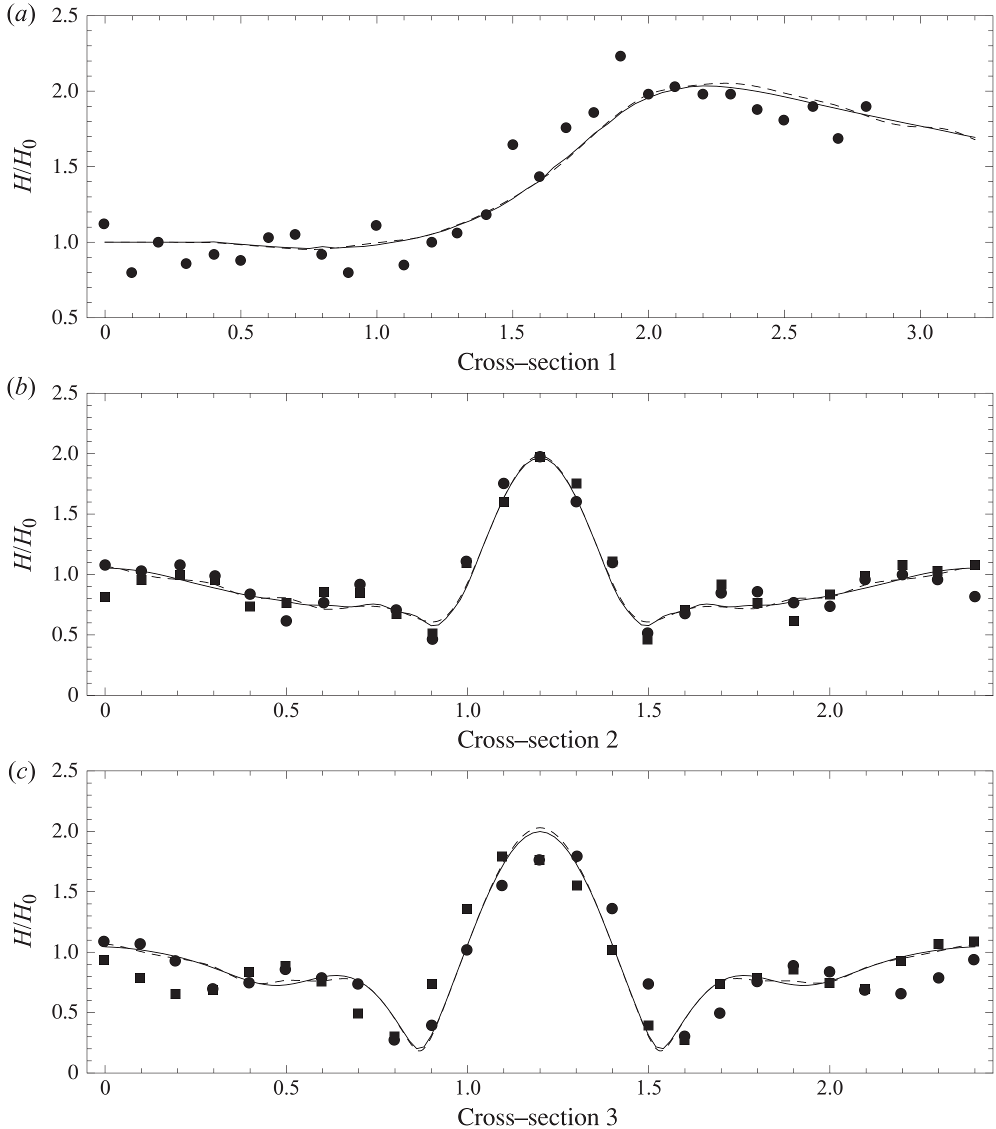

FIGURE 3. The normalized wave height for the experiment of Ito & Tanimoto (1972). The numerical results of the PMSE (solid) and the MMSE (dashed) are given together with the wave gauge measurements (solid circles/squares). The location of the cross-sections are given in figure 2.

accurate analytical solution and a wave tank experiment of three-dimensional nature. The results show very good agreements that reassure the use of this model for solving practical wave problems.

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